

# Symmetric Pseudospherical Surfaces I: General Theory

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*In Memory of Katsumi Nomizu*

**Abstract.** We apply the loop group method developed by Zakharov-Shabat [14], Terng-Uhlenbeck [9] and Toda [11] to the study of symmetries of pseudospherical surfaces (ps-surfaces) in  $\mathbb{R}^3$ . In this paper (part I) we consider the general theory, while in a second paper (part II) we will study special cases.

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## 1. Introduction

In this paper, we study symmetries of pseudospherical surfaces (ps-surfaces, i.e. surfaces with Gauss curvature  $K = -1$ ) in  $\mathbb{R}^3$  via the loop group method developed by Zakharov-Shabat [14], Terng-Uhlenbeck [9] and Toda [11]. One of the motivations for studying symmetries is to develop a theory for non-finite-type ps-surfaces. (A rather complete investigation of ps-surfaces of finite-type is Melko-Sterling [7].) In particular, using methods in Part I we will exhibit examples with discrete rotational symmetry about an axis. These examples contain points which have properties similar to umbilic points. We believe these examples will help to develop a theory of ps-surfaces of non-finite type.

In §2 of the paper we review the main results of Toda's algorithm as it has been used computationally for several years. First, we discuss the 1:1 correspondence (up to rigid motions) between ps-surfaces parametrized by asymptotic lines and pairs of *normalized potentials*. In preparation for understanding the relationships between symmetries at various levels, we review the precise correspondence between four levels of description for a ps-surface: the immersion  $f : D \rightarrow \mathbb{R}^3$  itself, the extended orthonormal frame  $F$ , the extended  $SU(2)$ -valued frame  $U$ , and the

normalized potential pair. The main results in this section are the construction of normalized potentials in (2.14) and (2.16), and the converse in Theorem 2.7. We also address the questions of uniqueness and differentiability, and introduce generalized potentials.

In §3 we study symmetries of ps-surfaces, frames and potentials. Our study is similar to that of Dorfmeister and Haak's study of symmetries of constant mean curvature surfaces [2],[3]. Our basic assumption is that there is a rigid motion  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and a diffeomorphism  $\gamma : D \rightarrow D$  such that

$$f \circ \gamma = R \circ f.$$

In particular we address the issue of how group actions on the surfaces relate to group actions on the space of general potentials. The main results in this section are Propositions 3.3 and 3.5.

In the second paper [4] we will study special cases, both old and new, including symmetries via the fundamental group or rotational invariance. New examples include several with discrete rotational symmetry. One such example is shown in Figures 1 and 2 of this paper.

## 2. Loop Groups and Pseudospherical Surfaces

Here, we want to summarize the loop group method for constructing ps-surfaces. We aim to give a compact exposition, so some proofs will be omitted.

### 2.1. Ps-Surface to Darboux Frame

We begin by reviewing some well-known facts about ps-surfaces, beginning with the fact that the asymptotic lines form a Chebyshev net, and the angle between them satisfies the sine-Gordon equation:

**Theorem 2.1.** *Let  $f : D \rightarrow \mathbb{R}^3$  be an oriented immersed ps-surface. Near any point of  $D$ , there are coordinates  $x$  and  $y$  such that  $\partial f / \partial x$  and  $\partial f / \partial y$  are unit vectors and asymptotic directions, and  $\partial f / \partial x \times \partial f / \partial y$  agrees with the orientation. Then the counterclockwise angle  $\phi$  from  $\partial f / \partial x$  to  $\partial f / \partial y$  satisfies the sine-Gordon equation*

$$\phi_{xy} = \sin \phi. \quad (2.1)$$

Let  $\theta = \phi/2$ . If we define the Darboux frame<sup>1</sup> (see figure below)

$$e_1 = \frac{1}{2} \sec(\theta)(\partial f / \partial x + \partial f / \partial y), \quad e_2 = \frac{1}{2} \csc(\theta)(\partial f / \partial y - \partial f / \partial x), \quad e_3 = e_1 \times e_2,$$

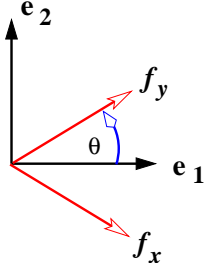
then the orthogonal matrix  $\tilde{F}$  whose columns are  $e_1, e_2, e_3$  satisfies

$$\frac{\partial \tilde{F}}{\partial x} = \tilde{F} \begin{bmatrix} 0 & \theta_x & -\sin \theta \\ -\theta_x & 0 & -\cos \theta \\ \sin \theta & \cos \theta & 0 \end{bmatrix}, \quad \frac{\partial \tilde{F}}{\partial y} = \tilde{F} \begin{bmatrix} 0 & -\theta_y & -\sin \theta \\ \theta_y & 0 & \cos \theta \\ \sin \theta & -\cos \theta & 0 \end{bmatrix}. \quad (2.2)$$

For this calculation, see §6.4 in [6].

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<sup>1</sup>That is, a moving frame along the surface where the first two vectors are principal directions.



We also have the converse:

**Theorem 2.2.** *Let  $D$  be a simply connected open set in  $\mathbb{R}^2$  and let  $\tilde{F}(x, y)$  be an  $SO(3)$ -valued function on  $D$  that satisfies (2.2) for some smooth function  $\theta(x, y)$  on  $D$ . Then  $\phi = 2\theta$  satisfies the sine-Gordon equation, and there is a map  $f : D \rightarrow \mathbb{R}^3$  which is an immersion at points where  $\sin \phi \neq 0$ , whose image is a ps-surface with Darboux frame given by the columns of  $\tilde{F}$ .*

## 2.2. Darboux Frame to Extended Frame

Rather than using a Darboux frame, it will be more convenient to use a frame that includes the unit vector  $\partial f / \partial x$ . Accordingly, we let  $F$  denote the frame obtained by rotating the first two vectors of the Darboux frame  $\tilde{F}$  through the clockwise angle  $\theta$ , so that

$$F = \tilde{F} \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\frac{\partial F}{\partial x} = F \begin{bmatrix} 0 & \phi_x & 0 \\ -\phi_x & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \frac{\partial F}{\partial y} = F \begin{bmatrix} 0 & 0 & -\sin \phi \\ 0 & 0 & \cos \phi \\ \sin \phi & -\cos \phi & 0 \end{bmatrix}. \quad (2.3)$$

The sine-Gordon equation for  $\phi$  is derived as the compatibility condition for the overdetermined system (2.3), by setting  $\partial(F_x)/\partial y = \partial(F_y)/\partial x$ .

The sine-Gordon equation is invariant under the 1-parameter group of Lie symmetry transformations of the form  $T^\lambda(x, y) = (\lambda x, \lambda^{-1} y)$ ,  $\lambda > 0$ . Hence,  $F^\lambda = F \circ T^\lambda$  will satisfy an overdetermined system with the same compatibility condition; in fact,

$$\frac{\partial F^\lambda}{\partial x} = F^\lambda \begin{bmatrix} 0 & \phi_x & 0 \\ -\phi_x & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix}, \quad \frac{\partial F^\lambda}{\partial y} = \frac{1}{\lambda} F^\lambda \begin{bmatrix} 0 & 0 & -\sin \phi \\ 0 & 0 & \cos \phi \\ \sin \phi & -\cos \phi & 0 \end{bmatrix}. \quad (2.4)$$

For any fixed  $\lambda \in \mathbb{R}^+$ ,  $F^\lambda$  is an orthonormal frame for a ps-surface; these ps-surfaces make up an *associated family* of ps-surfaces, which includes the original ps-surface when  $\lambda = 1$ .

### 2.3. Lifting the Extended Frame

It will also be convenient to work with matrices in the Lie group  $SU(2)$  instead of  $SO(3)$ . Recall that we can identify  $\mathbb{R}^3$  with the Lie algebra  $\mathfrak{su}(2)$  in a way that the adjoint action of  $SU(2)$  corresponds to rotations in  $\mathbb{R}^3$ , with every rotation in  $SO(3)$  being realized by  $\text{Ad}(g)$  for two possible elements  $g \in SU(2)$ , differing by a minus sign. This gives a double cover  $\delta : SU(2) \rightarrow SO(3)$ . Provided that  $D$  is simply-connected, we can choose a well-defined lift of  $F^\lambda$  into  $SU(2)$ , and we let  $U : D \rightarrow SU(2)$  denote this mapping, which we will also refer to as the extended frame of the surface.

To specify the lifting, we fix an identification of the standard basis vectors  $\hat{i}, \hat{j}, \hat{k}$  for  $\mathbb{R}^3$  with matrices in  $\mathfrak{su}(2)$  given by

$$\hat{i} \leftrightarrow \frac{1}{2} \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \quad \hat{j} \leftrightarrow \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \hat{k} \leftrightarrow \frac{1}{2} \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}.$$

(This identification has the virtue that the cross-product in  $\mathbb{R}^3$  corresponds exactly to the Lie bracket in  $\mathfrak{su}(2)$ .) Let  $e_1, e_2, e_3$  denote the columns of  $F^\lambda = \delta(U)$ . Then

$$e_1 = U\hat{i}U^{-1}, \quad e_2 = U\hat{j}U^{-1}, \quad e_3 = U\hat{k}U^{-1}, \quad (2.5)$$

where we are now tacitly identifying the vectors  $e_i$  and  $\hat{i}, \hat{j}, \hat{k}$  with their matrix counterparts. We can use the differential equations satisfied by  $F^\lambda$  to deduce the components of

$$\omega = U^{-1}dU.$$

The system (2.4) implies that  $\partial e_1/\partial x = -\phi_x e_2$  and  $\partial e_3/\partial x = -\lambda e_2$ . Differentiating (2.5) shows that  $de_1 = U[\omega, \hat{i}]U^{-1}$  and  $de_3 = U[\omega, \hat{k}]U^{-1}$ . Thus, the  $dx$  coefficient in  $\omega$  must be  $-\phi_x \hat{k} + \lambda \hat{i}$ . Similarly, (2.4) implies that  $\partial e_1/\partial y = \lambda^{-1} \sin \phi e_3$  and  $\partial e_2/\partial y = -\lambda^{-1} \cos \phi e_3$ , so the  $dy$  coefficient in  $\omega$  must be

$$\lambda^{-1}(-\sin \phi \hat{j} - \cos \phi \hat{i}) = -\frac{1}{2\lambda} \begin{bmatrix} 0 & \mathbf{i} \cos \phi - \sin \phi \\ \mathbf{i} \cos \phi + \sin \phi & 0 \end{bmatrix} = -\frac{\mathbf{i}}{2\lambda} \begin{bmatrix} 0 & e^{\mathbf{i}\phi} \\ e^{-\mathbf{i}\phi} & 0 \end{bmatrix}.$$

Thus,  $U$  satisfies

$$\frac{\partial U}{\partial x} = \frac{\mathbf{i}}{2} U \begin{bmatrix} -\phi_x & \lambda \\ \lambda & \phi_x \end{bmatrix}, \quad \frac{\partial U}{\partial y} = -\frac{\mathbf{i}}{2\lambda} U \begin{bmatrix} 0 & e^{\mathbf{i}\phi} \\ e^{-\mathbf{i}\phi} & 0 \end{bmatrix}. \quad (2.6)$$

The compatibility condition for this system is equivalent to requiring that  $\omega$ , as an  $\mathfrak{su}(2)$ -valued 1-form on  $D$ , satisfy the Maurer-Cartan equation

$$d\omega = -\omega \wedge \omega$$

for each  $\lambda$ . Notice that the  $\lambda$ -dependent parts of  $\omega$  are on the off-diagonal only. This means that  $\omega$  is a 1-form on  $D$  taking values in the loop algebra

$$\Lambda \mathfrak{su}(2) = \{X : \mathbb{R}^* \rightarrow \mathfrak{su}(2) \mid X(-\lambda) = \text{Ad}(\sigma_3) \cdot X(\lambda)\}, \quad \text{where } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.7)$$

Likewise,  $U$  is a map from  $D$  to the loop group

$$\Lambda SU(2) = \{g : \mathbb{R}^* \rightarrow SU(2) \mid g(-\lambda) = \text{Ad}(\sigma_3) \cdot g(\lambda)\}.$$

(loops satisfying the  $\text{Ad}(\sigma_3)$  condition are sometimes referred to as *twisted*.) We will be specifically interested in those subgroups, denoted by  $\tilde{\Lambda}\mathfrak{su}(2)$  and  $\tilde{\Lambda}SU(2)$  respectively, consisting of loops which extend to  $\mathbb{C}^*$  as analytic functions of  $\lambda$ . (Note, however, that such extensions will take values in  $\mathfrak{sl}(2, \mathbb{C})$  and  $SL(2, \mathbb{C})$  respectively.) In fact, the goal of the method is to recover such loops from analytic data specified along a pair of characteristic curves in  $D$ .

Within the group of loops that extend analytically to  $\mathbb{C}^*$ , we define subgroups of loops which extend to  $\lambda = 0$  or  $\lambda = \infty$ :

$$\begin{aligned}\Lambda^+ SU(2) &= \{g \in \tilde{\Lambda}SU(2) | g = g_0 + \lambda g_1 + \lambda^2 g_2 + \dots\} \\ \Lambda^- SU(2) &= \{g \in \tilde{\Lambda}SU(2) | g = g_0 + \lambda^{-1} g_1 + \lambda^{-2} g_2 + \dots\}\end{aligned}$$

Within these, we let  $\Lambda_*^+ SU(2)$  and  $\Lambda_*^- SU(2)$  be the subgroups of loops where  $g_0$  is the identity matrix.

#### 2.4. Extended Frame to Normalized Potential Pair

A key tool we will use is

**Theorem 2.3 (Birkhoff Decomposition [5], [11]).** *The multiplication maps*

$$\Lambda_*^+ SU(2) \times \Lambda^- SU(2) \rightarrow \tilde{\Lambda}SU(2), \quad \Lambda_*^- SU(2) \times \Lambda^+ SU(2) \rightarrow \tilde{\Lambda}SU(2)$$

*are diffeomorphisms.*

*Remark 2.4.* In general, the Birkhoff decomposition theorem asserts that the multiplication maps are diffeomorphisms onto an open dense subset, known as the **big cell**. However, it follows from the recent result of Brander [1] that in the case of compact semisimple Lie groups like  $SU(2)$ , the big cell is everything.

We apply both Birkhoff decompositions to  $U$ , giving

$$U = U_+^X V_- = U_-^Y V_+, \quad (2.8)$$

where  $U_+^X, U_-^Y \in \Lambda_*^\pm SU(2)$ ,  $V_-(x, y) \in \Lambda^- SU(2)$ , and  $V_+(x, y) \in \Lambda^+ SU(2)$ . The superscripts in  $U_+^X$  and  $V_-^Y$  are justified by the following important insight:

**Proposition 2.5.**  *$U_+^X$  does not depend on  $y$ , and  $U_-^Y$  does not depend on  $x$ .*

*Proof.* From (2.6), it follows that  $\omega = U^{-1}dU$  has the form

$$\omega = A dx + B dy, \quad A = A_0 + \lambda A_1, \quad B = B_1 \lambda^{-1},$$

where  $A_0, A_1, B_1$  depend only on  $x$  and  $y$ . Differentiating  $U_+^X = U(V_-)^{-1}$  gives

$$(U_+^X)^{-1} dU_+^X = V_- (A dx + B dy) (V_-)^{-1} - dV_- (V_-)^{-1}. \quad (2.9)$$

The left-hand side contains only positive powers of  $\lambda$ , while only the  $dx$  term on the right contains such powers, so it follows that  $U_+^X$  depends on  $x$  and  $\lambda$  only. A similar argument shows that  $U_-^Y$  depends on  $y$  and  $\lambda$  only.  $\square$

For the rest of this section we will assume that the domain  $D$  on which  $U(x, y)$  is defined contains the origin, and that  $U(x, y)$  satisfies the following initial condition.

$$U(0, 0; \lambda) = I \quad \forall \lambda. \quad (2.10)$$

Note that, by the uniqueness of the Birkhoff splitting, this implies that  $U_-^Y, U_+^X, V_-, V_+$  are also equal to the identity matrix when  $x = y = 0$ .

With these conditions in hand, we can determine the differential equations for  $U_+^X$  and  $U_-^Y$ . First, the coefficients of non-positive powers of  $\lambda$  on the right-hand side of (2.9) must vanish, and hence

$$\frac{\partial V_-}{\partial y} = V_- B.$$

Letting  $V_- = V_{-0}(x, y) + O(\lambda^{-1})$ , we find that  $\partial V_{-0}/\partial y = 0$ . Taking coefficients of positive powers of  $\lambda$  in (2.9) now gives

$$(U_+^X)^{-1} \frac{\partial U_+^X}{\partial x} = \lambda V_{-0} A_1 (V_{-0})^{-1}. \quad (2.11)$$

Next, we must determine  $V_{-0}(x)$ . Restricting (2.9) to the line  $y = 0$  gives

$$(U_+^X)^{-1} \frac{\partial U_+^X}{\partial x} = V_- A (V_-)^{-1} - \frac{\partial V_-}{\partial x} (V_-)^{-1} \Big|_{y=0}. \quad (2.12)$$

Again, the left-hand side contains only positive powers of  $\lambda$ . Taking the  $\lambda^0$  coefficient in (2.12) gives an expression for  $\partial V_{-0}/\partial x$  which *a priori* involves products of  $A_1$  with the coefficient of  $\lambda^{-1}$  in  $V_-$ . To eliminate such terms, we apply the following

**Lemma 2.6.** *Suppose  $P(t) \in \Lambda^\pm SU(2)$ ,  $Q(t) \in \Lambda^\mp SU(2)$  and  $R(t) \in \Lambda^\pm \mathfrak{su}(2)$  satisfy*

$$P^{-1} \frac{\partial P}{\partial t} = QRQ^{-1} - \frac{\partial Q}{\partial t} Q^{-1}$$

*on some  $t$ -interval containing  $t_0$ , and  $P(t_0) = Q(t_0) = I$ . Then  $Q(t)$  has no  $\lambda$ -dependence.*

*Proof.* For simplicity, take the upper sign in the hypotheses, the proof for the lower sign being identical in form. Let  $S(t) \in \Lambda^+ SU(2)$  satisfy

$$\frac{\partial S}{\partial t} = SR, \quad S(t_0) = I.$$

Then

$$(SQ^{-1})^{-1} \frac{\partial (SQ^{-1})}{\partial t} = QRQ^{-1} - \frac{\partial Q}{\partial t} Q^{-1},$$

so that  $SQ^{-1}$  satisfies the same differential equation, as a function of  $t$ , that  $P$  does. Since  $SQ^{-1}$  also coincides with  $P$  when  $t = t_0$ , it follows that  $P = SQ^{-1}$  for all  $t$ -values in the interval. Hence  $Q = P^{-1}S \in \Lambda^+ SU(2)$ , and it follows that  $Q$  has no  $\lambda$ -dependence.  $\square$

Using  $t = x$ ,  $P = U_+^X$ ,  $Q = V_-|_{y=0}$  and  $R = A|_{y=0}$  in the lemma, we conclude that the restriction of  $V_-$  to the  $x$ -axis has no  $\lambda$ -dependence. Thus, we can replace  $V_-$  with  $V_{-0}$  in (2.12). Taking the  $\lambda^0$  coefficient in that equation and using 2.6 to get  $A_0$  now gives

$$\frac{\partial V_{-0}}{\partial x} = V_{-0}A_0 = \frac{i}{2}V_{-0} \begin{bmatrix} -\phi_x(x, 0) & 0 \\ 0 & \phi_x(x, 0) \end{bmatrix}.$$

Using the initial condition  $V_{-0}(0) = I$ , we obtain

$$V_{-0}(x) = \begin{bmatrix} e^{-i\alpha(x)/2} & 0 \\ 0 & e^{i\alpha(x)/2} \end{bmatrix} \quad (2.13)$$

where we take  $\alpha(x) = \phi(x, 0) - \phi(0, 0)$ . Substituting in (2.11) gives

$$(U_+^X)^{-1} \frac{\partial U_+^X}{\partial x} = \frac{i}{2}\lambda \begin{bmatrix} 0 & e^{-i\alpha(x)} \\ e^{i\alpha(x)} & 0 \end{bmatrix} =: \eta_+^X. \quad (2.14)$$

Similarly, differentiating  $U_-^Y = U(V_+)^{-1}$  gives

$$(U_-^Y)^{-1} dU_-^Y = V_+(A dx + B dy)(V_+)^{-1} - dV_+(V_+)^{-1}.$$

We restrict this equation to the  $y$ -axis, giving

$$(U_-^Y)^{-1} \frac{\partial U_-^Y}{\partial y} = V_+B(V_+)^{-1} - \frac{\partial V_+}{\partial y}(V_+)^{-1}. \quad (2.15)$$

Using Lemma 2.6 with  $t = y$ ,  $P = U_-$  and  $Q = V_+$ , we conclude that the coefficients of all the positive powers of  $\lambda$  in  $V_+$  vanish along the  $y$ -axis. (However, the coefficient of  $\lambda^0$  in  $V_+$  will depend on both  $x$  and  $y$ .) Then, examining the  $\lambda^0$  coefficient in (2.15) shows that, along the  $y$ -axis  $V_+$  is constant and equal to the identity matrix. Then

$$(U_-^Y)^{-1} \frac{\partial U_-^Y}{\partial y} = -\frac{i}{2\lambda} \begin{bmatrix} 0 & e^{i\beta(y)} \\ e^{-i\beta(y)} & 0 \end{bmatrix} =: \eta_-^Y \quad (2.16)$$

for  $\beta(y) = \phi(0, y)$ .

We will refer to  $\eta_+^X(x, \lambda)$  in (2.14) and  $\eta_-^Y(y, \lambda)$  in (2.16) as a pair of *normalized potentials*, by analogy with holomorphic potentials that are determined by the loops associated to constant mean curvature surfaces. We remark the formulas expressing the potentials in terms of  $\phi$ , which agree with those of Toda ([12], equations (23) and (24)), are analogues of Wu's formula for CMC surfaces and their associated harmonic maps [13].

## 2.5. Normalized Potentials to Ps-Surface

Just as holomorphic potentials can be used to reconstruct constant mean curvature surfaces, we can use normalized potentials to reconstruct ps-surfaces. To see how this works, suppose that  $V_- = V_{-0}T_-$  for  $T_-(x, y) \in \Lambda_*^- SU(2)$ . Then (2.8) implies that

$$(U_-^Y)^{-1} U_+^X V_{-0} = V_+(T_-)^{-1}.$$

The left-hand side is determined by the normalized potentials (i.e., by  $\alpha(x)$  and  $\beta(y)$ ), while the right-hand side is a Birkhoff splitting (albeit, with the  $\Lambda_*^-$  piece as the second factor). Thus, given  $\alpha(x)$  and  $\beta(y)$  and applying the splitting, we determine  $T_-(x, y)$ , and thus construct a loop satisfying (2.8) by setting  $U = U_+^X V_{-0} T_-$ .

More formally, we have the following

**Theorem 2.7.** *Let functions  $\alpha(x)$  and  $\beta(y)$  be defined on intervals  $D_1, D_2 \subset \mathbb{R}$  containing zero, satisfying  $\alpha(0) = 0$ . Let  $V_{-0}(x)$  be given by (2.13), and let  $U_+^X(x), U_-^Y(y)$  satisfy (2.14) and (2.16), respectively, with  $U_+^X(0) = I$  and  $U_-^Y(0) = I$ . Let  $D = D_1 \times D_2$ , and let*

$$(U_-^Y)^{-1} U_+^X V_{-0} = V_+(T_-)^{-1}$$

*be its Birkhoff decomposition, for  $V_+(x, y) \in \Lambda^+ SU(2)$  and  $T_-(x, y) \in \Lambda_*^- SU(2)$ . Then*

$$U = U_+^X V_{-0} T_- = U_-^Y V_+$$

*satisfies (2.10) and also satisfies (2.6) for a function  $\phi(x, y)$  on  $D$  such that*

$$\phi(x, 0) = \alpha(x) + \beta(0), \quad \phi(0, y) = \beta(y).$$

Of course, it follows that  $\phi(x, y)$  satisfies the sine-Gordon equation (2.1). Moreover, the *Sym formula*:

$$f(x, y; \lambda) = \frac{\partial U}{\partial \log \lambda} U^{-1} \quad (2.17)$$

gives a family of pseudospherical surfaces which, for each value of  $\lambda$ , have  $U(x, y; \lambda)$  as extended frame. (Note, however, that  $f$  may fail to be an immersion at some points.)

*Proof of Theorem 2.7.* Let  $V_- = V_{-0} T_-$ . Differentiating  $U_-^Y V_+ = U_+^X V_-$  with respect to  $x$  gives

$$(V_+)^{-1} \frac{\partial V_+}{\partial x} = (V_-)^{-1} \frac{\partial V_-}{\partial x} + (V_-)^{-1} \left( (U_+^X)^{-1} \frac{\partial U_+^X}{\partial x} \right) V_-. \quad (2.18)$$

The highest power of  $\lambda$  on the right-hand side is  $\lambda^1$  in the factor  $(U_+^X)^{-1} \frac{\partial U_+^X}{\partial x}$ , so  $U^{-1} \frac{\partial U}{\partial x} = (V_+)^{-1} \frac{\partial V_+}{\partial x}$  contains only  $\lambda^0$  and  $\lambda^1$ . Furthermore, because of the twisting condition, the  $\lambda^0$  term is diagonal and the  $\lambda^1$  term is off-diagonal. Substituting (2.14) into (2.18) gives

$$\begin{aligned} U^{-1} \frac{\partial U}{\partial x} &= \frac{i}{2} \begin{bmatrix} -\kappa(x, y) & 0 \\ 0 & \kappa(x, y) \end{bmatrix} + (V_{-0})^{-1} \frac{i}{2} \lambda \begin{bmatrix} 0 & e^{-i\alpha(x)} \\ e^{i\alpha(x)} & 0 \end{bmatrix} V_{-0} \\ &= \frac{i}{2} \begin{bmatrix} -\kappa(x, y) & \lambda \\ \lambda & \kappa(x, y) \end{bmatrix} \end{aligned} \quad (2.19)$$



for some function  $\kappa(x, y)$ . Equation (2.18) implies

$$(U_+^X)^{-1} \frac{\partial U_+^X}{\partial x} = V_- \left( (V_+)^{-1} \frac{\partial V_+}{\partial x} \right) (V_-)^{-1} - \frac{\partial V_-}{\partial x} (V_-)^{-1}.$$

As with (2.12), applying Lemma 2.6 to the last equation lets us conclude that the restriction of  $V_-$  to the  $x$ -axis has no negative powers of  $\lambda$ . Taking the  $\lambda^0$  coefficient of the restriction of (2.18) to the  $x$ -axis now gives

$$U^{-1} \frac{\partial U}{\partial x} \Big|_{y=0} = (V_{-0})^{-1} \frac{\partial V_{-0}}{\partial x} + O(\lambda) = \frac{i}{2} \begin{bmatrix} -\alpha'(x) & 0 \\ 0 & \alpha'(x) \end{bmatrix} + O(\lambda),$$

which implies that  $\kappa(x, 0) = \alpha'(x)$ .

Differentiating  $U_+^X V_- = U_-^Y V_+$  with respect to  $y$  gives

$$(V_-)^{-1} \frac{\partial V_-}{\partial y} = (V_+)^{-1} \frac{\partial V_+}{\partial y} + (V_+)^{-1} \left( (U_-^Y)^{-1} \frac{\partial U_-^Y}{\partial y} \right) V_+. \quad (2.20)$$

Because  $\partial V_{-0}/\partial y = 0$ , only negative powers of  $\lambda$  are present in  $U^{-1} \frac{\partial U}{\partial y} = (V_-)^{-1} \frac{\partial V_-}{\partial y}$ . Because the only power of  $\lambda$  in  $(U_-^Y)^{-1} \frac{\partial U_-^Y}{\partial y}$  is  $\lambda^{-1}$ , then this is also the only power in  $U^{-1} \frac{\partial U}{\partial y}$ . If we let  $V_+ = V_{+0} T_+$  for  $T_+ \in \Lambda_*^+ SU(2)$ , then  $V_{+0}$  is diagonal because of the twisting, and we can let

$$V_{+0}(x, y) = \begin{bmatrix} e^{i\psi(x, y)} & 0 \\ 0 & e^{-i\psi(x, y)} \end{bmatrix}$$

for some function  $\psi$ . Then

$$\begin{aligned} U^{-1} \frac{\partial U}{\partial y} &= (V_{+0})^{-1} \left( (U_-^Y)^{-1} \frac{\partial U_-^Y}{\partial y} \right) V_{+0} \\ &= -\frac{i}{2\lambda} \begin{bmatrix} 0 & e^{i(\beta(y) - 2\psi(x, y))} \\ e^{-i(\beta(y) - 2\psi(x, y))} & 0 \end{bmatrix}. \end{aligned} \quad (2.21)$$

In order to be consistent with the  $y$ -derivative in (2.6), we let

$$\phi(x, y) = \beta(y) - 2\psi(x, y).$$

Then the compatibility condition between (2.19) and (2.21) implies that  $\kappa = \phi_x$  and  $\phi$  satisfies the sine-Gordon equation.

Lastly, we want to show that  $\phi(0, y) = \beta(y)$ , which is equivalent to  $V_{+0}(0, y) = I$ . Letting  $B$  stand for the right-hand side in (2.21), then equation (2.20) can be rearranged to give

$$(U_-^Y)^{-1} \frac{\partial U_-^Y}{\partial y} = V_+ B (V_+)^{-1} - \frac{\partial V_+}{\partial y} (V_+)^{-1}.$$

As with (2.15), we restrict this equation to the  $y$ -axis, and apply Lemma 2.6 to conclude that  $V_+(0, y)$  has no positive powers of  $\lambda$ . We can replace  $V_+$  with  $V_{+0}$  in

the restriction of (2.21) to the  $y$ -axis, and taking the  $\lambda^0$  coefficient in that equation now shows that  $V_{+0}(0, y) = I$ .  $\square$

## 2.6. Uniqueness and Differentiability

Let  $\eta_f$  denote the unique normalized potential associated to an oriented ps-immersion  $f$  by the algorithm of §§2.1–2.4. (We use  $\eta$  as an abbreviation for a pair  $(\eta_-^Y, \eta_+^X)$  of normalized potentials.) Then if  $f$  and  $\hat{f}$  are two ps-immersions that differ by a rigid motion,  $\eta_f = \eta_{\hat{f}}$ .

Let  $\mathbf{f}[\eta]$  denote the ps-immersion associated to a potential pair  $\eta$  by the construction of Theorem 2.7 and the Sym formula (2.17) with  $\lambda = 1$ . Suppose  $f(x, y)$  is a ps-immersion with angle function  $\phi(x, y)$ . Then by Theorem 2.7 the angle function  $\tilde{\phi}$  for  $\mathbf{f}[\eta_f]$  agrees with  $\phi$  along the  $x$ - and  $y$ -axes. It follows by a theorem of Bianchi that  $\tilde{\phi} = \phi$ , and by a theorem of Enneper (cf. [10] Thm. 2.2.1 or [12] Cor. 2) that  $f$  and  $\mathbf{f}[\eta_f]$  differ by a rigid motion. Thus, there is a 1:1 relation, up to rigid motion, between (local) ps-immersions and normalized potentials.

So far, we have assumed that all objects under consideration are smooth, but we can be more precise about how degrees of differentiability behave under these correspondences. Let  $f$  be a ps-immersion which is  $C^n$  for  $n \geq 2$ . Then  $\phi$  is  $C^{n-1}$ , and therefore the potential pair given by (2.14) and (2.16) is  $C^{n-1}$  in  $x, y$  and analytic in  $\lambda \in \mathbb{C}^*$ .

Conversely, consider some potential pair  $\eta$  which is  $C^{n-1}$  in  $x, y$  for  $n \geq 3$  and analytic in  $\lambda \in \mathbb{C}^*$ . Then by solving (2.14) and (2.16) we obtain  $U_+^X, U_-^Y$  which are  $C^n$  in  $x, y$  and analytic in  $\lambda$ . The next step in our construction is the Birkhoff splitting of  $(U_-^Y)^{-1}U_+^X$  as  $V_+T_-^{-1}$ . Since the splitting is analytic [5],  $V_+, T_-$  are  $C^n$  in  $x, y$  and analytic in  $\lambda$ . Therefore  $U(x, y, \lambda)$  is  $C^n$  in  $x, y$  and analytic in  $\lambda$ . Now we use the Sym formula (2.17), which differentiates with respect to  $\log \lambda$ , to obtain a mapping  $f$  which is  $C^n$  in  $x, y$  and analytic in  $\lambda$ .

For each fixed value of  $\lambda$ ,  $f$  will be an immersion except at points where  $\sin \phi(x, y) = 0$ . (At such points, the surface could have singularities, either weakly regular points [7] or cone points, as we will see in Part II.) To see why these are the only points where  $f$  could fail to be an immersion, use the Sym formula and (2.6) to compute  $f_x = UV'U^{-1}$  and  $f_y = UW'U^{-1}$ , where

$$V = \frac{i}{2} \begin{bmatrix} -\phi_x & \lambda \\ \lambda & \phi_x \end{bmatrix}, \quad W = -\frac{i}{2\lambda} \begin{bmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{bmatrix}$$

and prime denotes differentiation with respect to  $\log \lambda$ . From the explicit expression for  $V'$  and  $W'$  it is clear that  $f_x$  and  $f_y$  are linearly dependent if and only if  $\sin \phi = 0$ .

## 2.7. Generalized Potentials

In §§2.1–2.4 we have reviewed the procedure which associates with every ps-surface a pair of potentials  $\eta = (\eta_+^X(x, \lambda), \eta_-^Y(y, \lambda))$  given by (2.14) and (2.16). In these potentials the dependence on  $\lambda$  is quite simple:  $\eta_+^X = \lambda \xi_+^X$  and  $\eta_-^Y = \lambda^{-1} \xi_-^Y$  for a

pair of off-diagonal skew-Hermitian  $2 \times 2$  matrices  $\xi_+^X$  and  $\xi_-^Y$  depending only on  $x$  and  $y$  respectively.

It is also possible to construct ps-surfaces from potentials which are more general, in the sense that more powers of  $\lambda$  are involved. To indicate that more than one power of  $\lambda$  may be involved we will use the notation

$$\begin{aligned}\eta_{\#}^X(x, \lambda) &= \lambda \xi_1^X(x) + \lambda^0 \xi_0^X(x) + \mathcal{O}(\lambda^{-1}), \\ \eta_b^Y(y, \lambda) &= \lambda^{-1} \xi_1^Y(y) + \lambda^0 \xi_0^Y(y) + \mathcal{O}(\lambda),\end{aligned}\tag{2.22}$$

Note in particular that  $\eta_{\#}^X$  is not necessarily in  $\Lambda^+ SU(2)$  and  $\eta_b^Y$  is not necessarily in  $\Lambda^- SU(2)$ .

**Definition 2.8.** We say that  $\mathfrak{su}(2)$ -valued loops  $\eta_{\#}^X(x)$  and  $\eta_b^Y(y)$  are *generalized potentials for a pseudospherical surface* if the extended frame  $U$  of the surface satisfies splittings of the form

$$U = G_{\#}^X L_- = G_b^Y L_+, \tag{2.23}$$

for some  $L_{\pm}(x, y) \in \Lambda^{\pm} SU(2)$  and  $SU(2)$ -valued matrices  $G_{\#}^X(x, \lambda), G_b^Y(y, \lambda)$  which have  $\eta_{\#}^X, \eta_b^Y$  as their respective Maurer-Cartan matrices, i.e.,

$$\frac{\partial G_{\#}^X}{\partial x} = G_{\#}^X \eta_{\#}^X, \quad \frac{\partial G_b^Y}{\partial y} = G_b^Y \eta_b^Y. \tag{2.24}$$

Note that we do not impose the initial condition (2.10) on  $U$  or the other factors in (2.23).

**Theorem 2.9.** Let  $D_1, D_2$  be intervals on the real line and let  $\eta_{\#}^X(x)$  and  $\eta_b^Y(y)$  be two loops in  $\mathfrak{sl}(2, \mathbb{C})$ , defined for  $x \in D_1$  and  $y \in D_2$  respectively, which are smooth in  $x$  and  $y$ , analytic in  $\lambda \in \mathbb{C}^*$ , and which have expansions of the form (2.22) with nonvanishing  $\xi_1^X$  and  $\xi_1^Y$ . We also assume that these loops are  $\mathfrak{su}(2)$ -valued when  $\lambda$  is real, and that they satisfy the twisting condition given in (2.7). Then  $\eta_{\#}^X, \eta_b^Y$  are generalized potentials for a pseudospherical immersion defined for  $(x, y) \in D = D_1 \times D_2$ .

Note that the twisting condition implies that the coefficients of even powers of  $\lambda$  take value in  $\mathfrak{h}_0$ , the diagonal subalgebra of  $\mathfrak{su}(2)$ , while the coefficients of odd powers take value in  $\mathfrak{h}_1$ , the subspace of off-diagonal matrices.

*Proof.* Let  $G_{\#}^X, G_b^Y$  be as in (2.24). Suppose that

$$\xi_1^X = \frac{i}{2} a(x) \begin{pmatrix} 0 & e^{-i\alpha(x)} \\ e^{i\alpha(x)} & 0 \end{pmatrix}$$

for a nonzero real function  $a(x)$ , and let

$$T^X = \begin{pmatrix} e^{-i\alpha(x)/2} & 0 \\ 0 & e^{i\alpha(x)/2} \end{pmatrix}.$$

Construct the Birkhoff splitting

$$(G_b^Y)^{-1} G_{\sharp}^X T^X = L_+ (L_-)^{-1}, \quad (2.25)$$

where  $L_+(x, y) \in \Lambda^+ SU(2)$  and  $L_-(x, y) \in \Lambda_-^* SU(2)$ , and let

$$U = G_{\sharp}^X T^X L_- = G_b^Y L_+.$$

We compute

$$U^{-1} \frac{\partial U}{\partial y} = (L_-)^{-1} \frac{\partial L_-}{\partial y} = (L_+)^{-1} \left( \eta_b^Y L_+ + \frac{\partial L_+}{\partial y} \right).$$

The right-hand side has at most one negative power of  $\lambda$ , while the middle member is in  $\Lambda_-^* \mathfrak{su}(2)$ . Thus, if we write  $L_+ = L_{+0}(x, y) + \mathcal{O}(\lambda)$ , then  $U^{-1} \frac{\partial U}{\partial y} = (L_{+0})^{-1} \xi_1^Y L_{+0}$  and is  $\mathfrak{h}_1$ -valued. Similarly, we compute

$$U^{-1} \frac{\partial U}{\partial x} = (L_+)^{-1} \frac{\partial L_+}{\partial x} = (\tilde{L}_-)^{-1} \left( \eta_{\sharp}^X \tilde{L}_- + \frac{\partial \tilde{L}_-}{\partial x} \right),$$

where  $\tilde{L}_- = T^X L_-$ . The right-hand side has at most one positive power of  $\lambda$ , while the middle member has only non-negative powers of  $\lambda$ , so we have

$$U^{-1} \frac{\partial U}{\partial x} = A_0 + \lambda A_1,$$

where  $A_0$  depends on  $L_+$ , but

$$A_1 = (T^X)^{-1} \xi_1^X T^X = \frac{i}{2} a(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Suppose that

$$\xi_1^Y = \frac{i}{2} \begin{pmatrix} 0 & \rho(y) \\ \rho(y) & 0 \end{pmatrix}, \quad L_{+0} = \begin{pmatrix} e^{i\psi(x, y)} & 0 \\ 0 & e^{-i\psi(x, y)} \end{pmatrix}.$$

Then  $U$  satisfies the linear system

$$\frac{\partial U}{\partial x} = \frac{i}{2} a(x) U \begin{pmatrix} -\kappa(x, y) & \lambda \\ \lambda & \kappa(x, y) \end{pmatrix}, \quad \frac{\partial U}{\partial y} = \frac{i}{2\lambda} U \begin{pmatrix} 0 & \rho e^{-2i\psi} \\ \rho e^{2i\psi} & 0 \end{pmatrix}, \quad (2.26)$$

for some function  $\kappa$ . Set  $\rho(y) = -b(y)e^{i\beta(y)}$  for a real-valued function  $b(y) > 0$ . Then the second equation in (2.26) agrees with the second equation in (2.6), with the speed  $b(y)$  inserted, for

$$\phi(x, y) = \beta(y) - 2\psi(x, y).$$

Then the compatibility condition for (2.26) implies that  $\kappa = (a(x))^{-1} \partial\phi/\partial x$ . Note that  $\phi$  satisfies a version of the sine-Gordon equation:

$$\phi_{xy} = a(x)b(y) \sin \phi.$$

The existence of a pseudospherical surface with extended frame  $U$  now follows by integration.  $\square$

**Theorem 2.10.** Assume  $\eta = (\eta_{\sharp}^X(x), \eta_b^Y(y))$  is a generalized potential pair of some ps-surface, satisfying the hypotheses of Theorem 2.9. Given any  $Q_-^X(x, \lambda) \in \Lambda^- SU(2)$  and  $Q_+^Y(y, \lambda) \in \Lambda^+ SU(2)$  defined for  $x \in D_1$  and  $y \in D_2$ , let

$$\begin{aligned}\tilde{\eta}_{\sharp}^X &= (Q_-^X)^{-1} \eta_{\sharp}^X Q_-^X + (Q_-^X)^{-1} \frac{d}{dx} Q_-^X, \\ \tilde{\eta}_b^Y &= (Q_+^Y)^{-1} \eta_b^Y Q_+^Y + (Q_+^Y)^{-1} \frac{d}{dy} Q_+^Y.\end{aligned}\tag{2.27}$$

Then  $\tilde{\eta} = (\tilde{\eta}_{\sharp}^X, \tilde{\eta}_b^Y)$  is a generalized potential for the same surface. Moreover, any two generalized potentials for the same surface are related by some gauge transformation of the form (2.27).

*Proof.* To prove the first assertion, note that  $\tilde{G}_{\sharp}^X = G_{\sharp}^X Q_-^X$  and  $\tilde{G}_b^Y = G_b^Y Q_+^Y$  satisfy (2.24) with  $\eta$  replaced by  $\tilde{\eta}$ . Then the corresponding splitting in (2.25) is satisfied by  $\tilde{L}^+ = (Q_+^Y)^{-1} L^+$  and  $\tilde{L}_- = (T^X)^{-1} Q_{-0}^X (Q_-^X)^{-1} T^X L_-$ , where  $Q_{-0}^X$  denotes the term in  $Q_-^X$  of order zero in  $\lambda$ . Then  $\tilde{U} = \tilde{G}_b^Y \tilde{L}_+ = U$ .

For the proof of the second assertion we recall from §2.6 that normalized potentials are in 1-1 relation to local ps-immersions up to rigid motions. Therefore, if two potentials  $\tilde{\eta}$  and  $\eta$  induce the same surface, then, after fixing the frame at some basepoint, we can assume that they induce the same extended frame. Comparing now the defining equations for the extended frame we obtain (for example)

$$U = G_{\sharp}^X L_- = \tilde{G}_{\sharp}^X \tilde{L}_-$$

where

$$\frac{\partial G_{\sharp}^X}{\partial x} = G_{\sharp}^X \eta_{\sharp}^X, \quad \frac{\partial \tilde{G}_{\sharp}^X}{\partial x} = \tilde{G}_{\sharp}^X \tilde{\eta}_{\sharp}^X.$$

Letting  $Q_-^X = L_- \tilde{L}_-^{-1}|_{y=0}$ , we have  $\tilde{G}_{\sharp}^X = G_{\sharp}^X Q_-^X$ , and thus

$$\tilde{\eta}_{\sharp}^X = (\tilde{G}_{\sharp}^X)^{-1} \frac{\partial \tilde{G}_{\sharp}^X}{\partial x} = (Q_-^X)^{-1} \eta_{\sharp}^X Q_-^X + (Q_-^X)^{-1} \frac{\partial Q_-^X}{\partial x}. \quad \square$$

The result above implies a somewhat surprising but useful

**Corollary 2.11.** Assume that  $U(x, y; \lambda)$  is the extended frame of some ps-surface. Then

$$\eta_{\sharp}^X(x, \lambda) := U^{-1} \frac{\partial U}{\partial x} \Big|_{y=x}, \quad \eta_b^Y(y, \lambda) := U^{-1} \frac{\partial U}{\partial y} \Big|_{x=y}$$

are generalized potentials for the same surface.

*Proof.* We can show that  $(\eta_{\sharp}^X, \eta_b^Y)$  differ from the normalized potential  $(\eta_+^X, \eta_-^Y)$  of  $U$  by a gauge transformation (2.27). For example, from (2.8) and (2.14) we have

$$U^{-1} \frac{\partial U}{\partial x} = (V_-)^{-1} \eta_+^X V_- + (V_-) \frac{\partial V_-}{\partial x},$$

where  $V_-$  depends on  $x$  and  $y$ . Letting  $Q_-^X = V_-|_{y=x}$ , we have

$$\eta_{\sharp}^X = (Q_-^X)^{-1} \eta_+^X Q_-^X + (Q_-^X)^{-1} \frac{\partial Q_-^X}{\partial x}. \quad \square$$

### 3. Symmetric Ps-Surfaces

#### 3.1. Symmetric Ps-Surfaces and Frames

Let  $f : D \rightarrow \mathbb{R}^3$  be a ps-immersion, where  $D \subset \mathbb{R}^2$  is an open set. We assume that  $f$  is nondegenerate at each point in  $D$ , and that the  $x$ - and  $y$ -coordinate curves are asymptotic lines on the surface  $f(D)$  (not necessarily of unit speed). By convention, we associate an oriented orthonormal frame  $F(x, y)$  to  $f$  in a unique way, so that the first column of  $F$  is the unit vector in the direction of  $f_x$ , and the third column is the unit vector in the direction of the surface normal  $\mathbf{n} = f_x \times f_y$ .

The most natural notion of a symmetry of an immersion  $f$  seems to be that there exists some rigid motion  $R$  such that  $R \circ f(D) = f(D)$ . Around any point  $p_0$  in this subset there exists an open set  $U \subset D$  and map  $\gamma : U \rightarrow D$  such that  $f \circ \gamma(p) = R \circ f(p)$  and  $\gamma$  is a diffeomorphism onto its image. In many cases (and under natural assumptions, like completeness of the pullback metric on  $D$ ) such a  $\gamma$  exists globally. We therefore will use from now on the **basic assumption** that there is a rigid motion  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and a diffeomorphism  $\gamma : D \rightarrow D$  such that

$$f \circ \gamma = R \circ f. \quad (3.1)$$

**Proposition 3.1.** *Let  $R' \in O(3)$  be the linear part of  $R$ . Then there is a matrix  $K(x, y) \in O(3)$ , with  $\det K = \det R'$ , such that*

$$F \circ \gamma = R' F K. \quad (3.2)$$

Moreover, the Maurer-Cartan form  $K^{-1}dK$  takes value in the subalgebra

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \subset \mathfrak{so}(3).$$

In fact, the matrix  $K(x, y)$  can be calculated as follows: Let  $\epsilon = \pm 1$  be such that  $R'\mathbf{n} = \epsilon(\mathbf{n} \circ \gamma)$ , let  $J$  be the Jacobian of  $\gamma$ , and let  $Z$  be the  $2 \times 2$  upper triangular matrix such that

$$[f_x, f_y] = F \begin{bmatrix} Z & \\ 0 & 0 \end{bmatrix}.$$

Then

$$K(x, y) = \begin{bmatrix} ZJ^{-1}(Z \circ \gamma)^{-1} & 0 \\ 0 & \epsilon \end{bmatrix}.$$

From now on, assume that  $R'$  and  $K(x, y)$  are  $SO(3)$ -valued, i.e., the rigid motion is proper.

### 3.2. Extended Frames and Monodromy

Note that  $\gamma$  preserves asymptotic coordinates. For, it follows from (3.1) that

$$(f \circ \gamma)_x = R' f_x, \quad (f \circ \gamma)_y = R' f_y, \quad (f \circ \gamma)_{xx} = R' f_{xx}.$$

Hence  $(f \circ \gamma)_x$  is an asymptotic vector if and only if  $\det[R' f_x, R' f_y, R' f_{xx}] = \det R' \det[f_x, f_y, f_{xx}] = 0$ , i.e.,  $f_x$  is an asymptotic vector. The same is true for  $(f \circ \gamma)_y$ . Thus,  $\gamma$  either preserves or switches the  $(x, y)$  coordinates.

**Proposition 3.2.** *Let  $F^\lambda$  be the extended frame for the given ps-surface. If  $\gamma$  takes  $x$ -coordinate curves to  $x$ -coordinate curves and  $y$ -coordinate curves to  $y$ -coordinate curves, then there is an  $SO(3)$ -valued function  $\chi(\lambda)$  such that*

$$F^\lambda \circ \gamma = \chi F^\lambda K, \quad (3.3)$$

and moreover  $K$  depends only on  $x$ . If  $\gamma$  switches the  $x$ - and  $y$ -coordinate curves, then we have

$$F^\lambda \circ \gamma = \chi F^{1/\lambda} K. \quad (3.4)$$

*Proof.* Let  $\omega^\lambda = (F^\lambda)^{-1} dF^\lambda$ , where  $\lambda$  is treated as a constant. From (2.4) we know that

$$\omega^\lambda = (\alpha_0 + \alpha_1 \lambda) dx + \beta_1 dy,$$

where  $\alpha_0$  takes value in  $\mathfrak{h}_0$  and  $\alpha_1, \beta_1$  in  $\mathfrak{h}_1$ .

Differentiating each side of (3.2) gives

$$\gamma^* \omega = K^{-1} \omega K + K^{-1} dK,$$

where  $\omega$  simply denotes the value of  $\omega^\lambda$  when  $\lambda = 1$ . Taking the  $\mathfrak{h}_0$  and  $\mathfrak{h}_1$  parts of each side in this equation, we have

$$\begin{aligned} \gamma^*(\alpha_0 dx) &= K^{-1} \alpha_0 K dx + K^{-1} dK, \\ \gamma^*(\alpha_1 dx + \beta_1 dy) &= K^{-1} (\alpha_1 dx + \beta_1 dy) K. \end{aligned} \quad (3.5)$$

1. Assume that  $\gamma$  preserves the set of  $x$ -coordinate curves. Then  $\gamma^* dx$  is a multiple of  $dx$ , and the second line in (3.5) implies that

$$\begin{aligned} \gamma^*(\alpha_1 dx) &= K^{-1} \alpha_1 K dx, \\ \gamma^*(\beta_1 dy) &= K^{-1} \beta_1 K dy. \end{aligned}$$

Multiplying these by powers of  $\lambda$  and combining with the first line in (3.5) gives

$$\gamma^* \omega^\lambda = K^{-1} \omega^\lambda K + K^{-1} dK.$$

The left-hand side is the Maurer-Cartan form of  $F^\lambda \circ \gamma$ , while the right-hand side is the Maurer-Cartan form of  $F^\lambda K$ . Then (3.3) follows by a standard theorem for maps into Lie groups (see, e.g., Theorem 10.18 in [8]). Note that we must assume that the domain  $D$  is connected to use this. The  $x$ -dependence of  $K$  follows from the first line of (3.5).

2. Instead, assume that  $\gamma$  exchanges the  $x$ - and  $y$ -coordinate curves. Then the second line in (3.5) implies that

$$\begin{aligned}\gamma^*(\alpha_1 dx) &= K^{-1}\beta_1 K dy, \\ \gamma^*(\beta_1 dy) &= K^{-1}\alpha_1 K dx.\end{aligned}$$

Multiplying these by powers of  $\lambda$  and combining with the first line in (3.5) gives

$$\gamma^*\omega^\lambda = K^{-1}\omega^{1/\lambda}K + K^{-1}dK.$$

Now (3.4) follows from this by the same argument as above.  $\square$

We lift the equation (3.3) up to  $SU(2)$ , whereupon  $K^{-1}dK$  takes value in the subalgebra  $\mathfrak{h}_0$  of diagonal matrices in  $\mathfrak{su}(2)$ . Then the lift  $U$  of  $F^\lambda$  to  $SU(2)$  satisfies

$$\frac{\partial U}{\partial x} = \frac{i}{2}U \begin{bmatrix} -\phi_x & a(x)\lambda \\ a(x)\lambda & \phi_x \end{bmatrix}, \quad \frac{\partial U}{\partial y} = -\frac{ib(y)}{2\lambda}U \begin{bmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{bmatrix}, \quad (3.6)$$

where  $a = |\partial f/\partial x|$  and  $b = |\partial f/\partial y|$  are the speeds in the  $x$ - and  $y$ -directions. We will write the lifted version of (3.3) as

$$U \circ \gamma = \chi U K \quad (3.7)$$

**Proposition 3.3.** *Assume that  $SU(2)$ -valued functions  $U(x, y; \lambda)$ ,  $K(x, y)$  and  $\chi(\lambda)$  satisfy (3.6) and (3.7) on  $D$ . Then  $\gamma$  takes  $x$ -coordinate curves to  $x$ -coordinate curves and  $y$ -coordinate curves to  $y$ -coordinate curves.*

*Proof.* We apply a Birkhoff splitting to  $U$ , yielding

$$U = U_+^X V_- = U_-^Y V_+, \quad (3.8)$$

where, by Prop. 2.5,  $U_+^X$  has no  $y$ -dependence and  $U_-^Y$  has no  $x$ -dependence. Substituting the last splitting into (3.7) and removing  $V_+$  factors from the left-hand side gives

$$U_-^Y \circ \gamma = \chi U_-^Y W_+, \quad \text{where } W_+ = V_+ K (V_+ \circ \gamma)^{-1}.$$

Let  $(\gamma_1(x, y), \gamma_2(x, y))$  be the components of  $\gamma$ . Differentiating the last equation with respect to  $x$  and canceling  $U_-^Y \circ \gamma$  from each side gives

$$\frac{\partial \gamma_2}{\partial x} (\eta_- \circ \gamma_2) = W_+^{-1} \frac{\partial W_+}{\partial x},$$

where  $\eta_-$  is as in (2.16). Thus, the left-hand side contains only negative powers of  $\lambda$ , while the right-hand side contains no negative powers of  $\lambda$ . Because  $\eta_-$  does not vanish, we must have  $\partial \gamma_2/\partial x = 0$  (and also  $\partial W_+/\partial x = 0$ ). A similar argument shows that  $\partial \gamma_1/\partial y = 0$ .  $\square$



### 3.3. Extended Frames and Equivariant Potentials

In this section we consider the implications of the symmetry assumption for potentials derived from splitting the extended frame. As in §2.7 we will consider more general splittings of the form

$$U = G_b^Y L_+ = G_\sharp^X L_-, \quad (3.9)$$

where  $G_\sharp^X$  and  $G_b^Y$  are assumed to have the property that  $\partial G_\sharp^X / \partial y = 0$ ,  $\partial G_b^Y / \partial x = 0$ , and  $L_\pm \in \Lambda^\pm SU(2)$ . As in §2.7, let  $\eta_\sharp^X$  and  $\eta_b^Y$  be the Maurer-Cartan matrices of  $G_\sharp^X$  and  $G_b^Y$  respectively, defined by (2.24), which are assumed to have expansions of the form (2.22).

**Lemma 3.4.** *Let  $U(x, y; \lambda)$ ,  $K(x, y)$  and  $\chi(\lambda)$  be as in Prop. 3.3 and assume we have splittings as in (3.9). Then these factors transform under  $\gamma$  according to*

$$G_b^Y \circ \gamma = \chi G_b^Y W_+^Y, \quad (3.10)$$

and

$$G_\sharp^X \circ \gamma = \chi G_\sharp^X W_-^X, \quad (3.11)$$

where  $W_-^X = L_- K (L_- \circ \gamma)^{-1}$  takes value in  $\Lambda^- SU(2)$  and has no  $y$ -dependence, and  $W_+^Y = L_+ K (L_+ \circ \gamma)^{-1}$  takes value in  $\Lambda^+ SU(2)$  and has no  $x$ -dependence. Then the generalized potentials of  $G_\sharp^X$ ,  $G_b^Y$  transform according to

$$(\eta_\sharp^X \circ \gamma) \frac{d\gamma_1}{dx} = (W_-^X)^{-1} \eta_\sharp^X W_-^X + (W_-^X)^{-1} \frac{dW_-^X}{dx}, \quad (3.12)$$

$$(\eta_b^Y \circ \gamma) \frac{d\gamma_2}{dy} = (W_+^Y)^{-1} \eta_b^Y W_+^Y + (W_+^Y)^{-1} \frac{dW_+^Y}{dy}, \quad (3.13)$$

where  $\gamma_1, \gamma_2$  are the components of  $\gamma$ .

*Proof.* Substituting the splitting  $U = G_b^Y L_+$  into (3.7) gives (3.10), and we similarly deduce (3.11). By Prop. 3.3,  $\gamma$  preserves  $x$ -coordinate and  $y$ -coordinate curves separately. Differentiating (3.10) and (3.11) then yields the last two equations.  $\square$

**Proposition 3.5.** *Let  $\gamma : D \rightarrow D$  separately preserve  $x$ - and  $y$ -coordinate lines. Assume that matrices  $G_\sharp^X(x, \lambda)$ ,  $G_b^Y(y, \lambda)$  satisfy (3.10) and (3.11) for some  $W_+^Y(y, \lambda)$  and  $W_-^X(x, \lambda)$  in  $\Lambda^+ SU(2)$  and  $\Lambda^- SU(2)$  respectively, and the same  $\chi(\lambda) \in SU(2)$ . Then there exist  $U, L_+, L_-$  satisfying (3.9) and a  $K(x, y) \in SU(2)$  such that (3.7) is satisfied for every  $\lambda$ .*

*Proof.* Apply a Birkhoff splitting to the product  $(G_b^Y)^{-1} G_\sharp^X$ , to find matrices  $L_\pm(x, y, \lambda) \in \Lambda^\pm SU(2)$  such that

$$L_+(L_-)^{-1} = (G_b^Y)^{-1} G_\sharp^X.$$

Composing with  $\gamma$  and applying the intertwining relations (3.10) and (3.11) yields

$$(L_+ \circ \gamma)(L_- \circ \gamma)^{-1} = W_+^X L_+(W_-^Y L_-)^{-1}.$$

Because we are not imposing that any factor lie in  $\Lambda_*^\pm SU(2)$ , the splitting factors are not unique; however the minus and plus factors on each side must agree up to a multiple that is independent of  $\lambda$ . Thus, we have

$$L_+ \circ \gamma = W_+^Y L_+ K(x, y), \quad L_- \circ \gamma = W_-^X L_- K(x, y)$$

for some  $K(x, y) \in SU(2)$ . From this it follows that  $U = G_b^Y L_+ = G_\sharp^X L_-$  satisfies (3.7). (Moreover, because all loops are assumed to be twisted by the automorphism of  $\mathfrak{su}(2)$  that preserves  $\mathfrak{h}_0$ , it follows that  $K^{-1}dK$  takes value in  $\mathfrak{h}_0$ .)  $\square$

Our starting point in constructing symmetric ps-surfaces will be to write down potentials satisfying (3.12) and (3.13). Note that these conditions mean that the pullbacks under  $\gamma$  of the 1-forms  $\eta_\sharp^X dx$  and  $\eta_b^Y dy$  are gauge-equivalent to themselves, under the gauges  $W_-^X$  and  $W_+^Y$  respectively. It is easiest to satisfy these if the gauge matrices  $W_+^Y$  and  $W_-^X$  are constant, as in the following.

### 3.4. Important Example

The general theory presented in this paper is a powerful tool for producing new classes of examples of ps-surfaces. The ps-cone points arising naturally in several of these examples help to clarify both the theory of discrete ps-surfaces and the theory of Lorentz umbilic points. In “Part II” we present these and other special cases of symmetry. Included among the special cases are all known examples and many new examples.

Here we will not derive but simply give the input data for an important example with discrete rotational symmetry. Let  $C(x) = (x - i)/(x + i)$  denote the Cayley transform, which maps the real line onto the unit circle. We choose

$$\eta_\sharp^X = (\lambda + \lambda^{-1}) \begin{bmatrix} 0 & p(x) \\ -p(x) & 0 \end{bmatrix}, \quad \eta_b^Y = (\lambda + \lambda^{-1}) \begin{bmatrix} 0 & p(y) \\ -p(y) & 0 \end{bmatrix},$$

where the function  $p$  which appears in both matrices is defined by

$$p(t) = \frac{d}{dt} \left( \frac{Q(w)}{w} \right), \quad \text{where } Q(w) = w^3 + w^{-3} \text{ and } w = C(t).$$

Then  $(\eta_\sharp^X, \eta_b^Y)$  satisfies the conditions (3.12), (3.13) for

$$W_-^X = W_+^Y = \begin{bmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{bmatrix}$$

and maps  $(\gamma_1, \gamma_2)$  of the  $x$  and  $y$  axes that each correspond under the Cayley transform to rotation by  $2\pi/3$  on the unit circle.

This choice of potential yields the ps-surface shown (from different viewpoints) in Figures 1 and 2. The surface is symmetric under a rotation (through an angle of approximately  $\pi/3$ ) around an axis which is perpendicular to the page. The figures show only a portion of the “bottom half” of the surface, the “top half” being given by reflecting through the plane perpendicular to the axis. This axis passes through a pseudospherical cone point. This point itself is the image of a line in the domain and could be thought of as a degenerate curvature line. It is

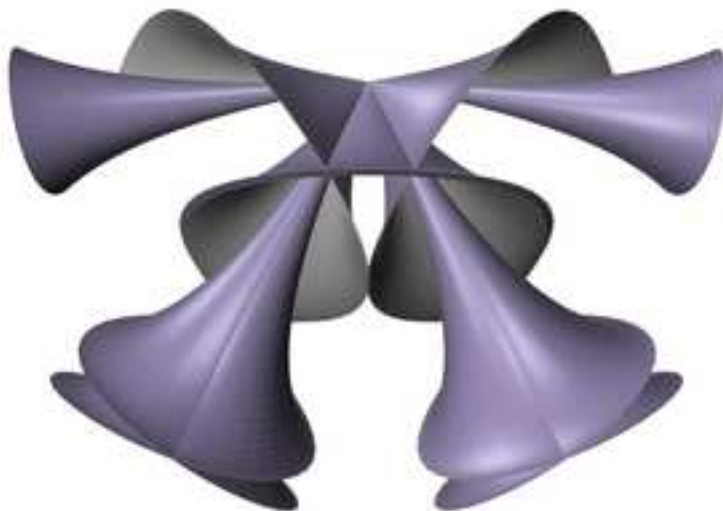


FIGURE 1. Generalized Amsler Surface

noteworthy that every asymptotic curve (in both families) goes through the cone point.

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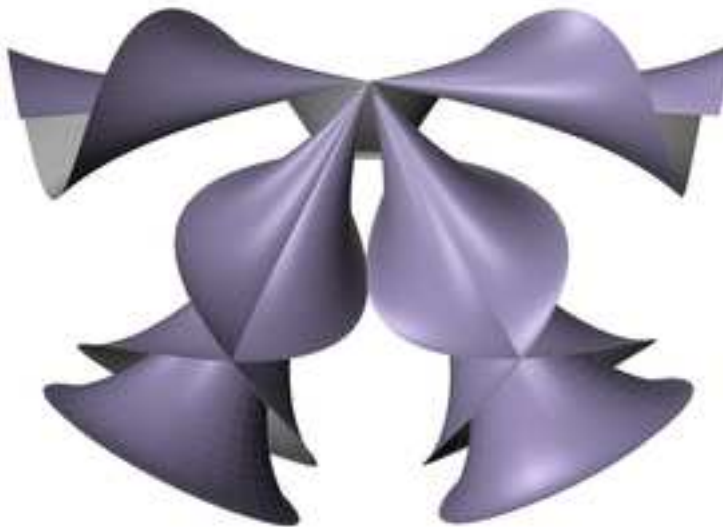


FIGURE 2. Generalized Amsler Surface

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